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Solving Problems on Finite Concrete Logics with the Help of a PC

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Three routines which may be used in the computer treatment of general finite concrete logics are described. The results for some particular finite concrete logics are presented.

1. INTRODUCTION

Treating finite OMPs with a computer is not an absolutely new area (see, e.g., Kalmbach, 1983; Navara, 1994). So far as I know, however, a computer investigation of finite concrete logics in the way we propose has not yet been undertaken. It is essential that we develop routines that enable us to examine finite concrete logics defined by generators. It is hoped that the routines will be helpful for many investigators in, e.g., constructing counteraxamples.

In Section 2, preliminaries are sketched on finite concrete logics. In Section 3, we describe three developed routines: GENER, BIPOLAR, and MINRE. In Section 4, the main section of this note, we discuss the results obtained with their help. We deal with concrete logics of the form $\Delta(\Lambda)$ yielded by finite-point collections on the plane which are closely related to the celebrated Zerbe–Gudder theorem (Zerbe and Gudder, 1985) and with representations as a concrete logic for the OMPs E_n whose Greechie diagrams are n-polygons with three atoms on each side.

2. PRELIMINARIES

Let Ω be a set. A *concrete logic* (c.l.) (cf. Gudder, 1979) on Ω is a collection $\mathscr{E}$ of subsets of Ω satisfying (1) $\Omega \in \mathscr{E}$, (2)$X \in \mathscr{E} \Rightarrow \Omega \backslash X \in \mathscr{E}$, (3) $X,Y \in \mathscr{E}$, $X \cap Y = \emptyset \Rightarrow X \cup Y \in \mathscr{E}$.

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Let E be an OMP with a full (= order-determining) set of two-valued finitely additive states. A *representation* (Sultanbekov, 1993) for E is an arbitrary c.l. isomorphic to E as an OMP. Let $(\Omega, \mathscr{E})$ be a representation for E. $\mathscr{E}$ is called *minimal* (Sultanbekov, 1993) provided that $\forall\ \omega \in \Omega\ \exists X, Y \in \mathscr{E}\ (X \cap Y = \{\omega\})$. Let Ω be a finite set and $\mathscr{E}$ a c.l. on Ω. The *polar* (cf. Ovchinnikov, 1994) $\mathscr{E}^{\circ}$ of $\mathscr{E}$ is defined as the set of all signed measures μ on the Boolean algebra of all subsets of Ω with $\mu(X) = 0$ for all $X \in \mathscr{E}$. The set $\mathscr{E}^{\circ\circ}$ of all $X \subset \Omega$ satisfying $\mu(X) = 0$ for every $\mu \in \mathscr{E}^{\circ}$ is obviously a c.l. on Ω which we call the *bipolar* of $\mathscr{E}$. $\mathscr{E}$ is called *closed* (Ovchinnikov, 1994) if $\mathscr{E} = \mathscr{E}^{\circ\circ}$.

Let $A_1, \ldots, A_m \subset \Omega$. The $A_1, \ldots, A_m$ are called *generators* for the least, with respect to inclusion, c.l. on Ω containing these. The c.l. is referred to as *generated* by $A_1, \ldots, A_m$.

Let E be an OMP. We call a *block* in E a maximal, with respect to inclusion, ortogonal set of atoms in E. (We acknowledge that this definition of a block is somewhat nonstandard.)

3. THE ROUTINES AND TESTS FOR THEM

Three routines in Turbo Pascal have been developed: GENER, BIPOLAR, and MINRE. In doing this, we managed to restrict ourselves to the integer operations alone.

GENER. Let $\Omega = \{1, \ldots, n\}$. In the data file, ***.dgn**, the user specifies the n, the number m of the generators, and the generators themselves as $n_1\ n_2 \ldots n_k$, $n_i \in \{1, \ldots, n\}$ each. In the results files, ***.atm** and ***.blk**, all atoms and all blocks of the generated c.l. are written. The processing time is also specified. GENER was tested with the Gudder–Marchand logics $\Sigma(L, s)$, examined in Ovchinnikov (1992).

Let $n = Ls$, where $s \geq 3$, $L \geq 2$, and let $\Sigma(L,s)$ be the c.l. on Ω with $\{0, 1, \ldots, L-1\} + k(\mathrm{mod}\, Ls)$, $k \in \{0, 1, \ldots, Ls-1\}$ as generators. For the following couples (L, s) the formulas s^L and $(s!)^{L-1}$ for the numbers of atoms and blocks in $\Sigma(L, s)$, respectively, resulting from Ovchinnikov (1992) have been confirmed: (2, 8), (3, 5), (4, 4), (5, 3), (6, 3), and (7, 3).

BIPOLAR. Given a c.l. $\mathscr{E}$ on Ω, the user should specify $\mathscr{E}^{\circ}$ (in the data file ***.dbp**) by defining the coefficients of the general solution of the corresponding homogeneous linear system multiplied by a suitable integer (which makes them all integer). In the results files, ***.atm** and ***.blk**, all atoms and all blocks in $\mathscr{E}^{\circ\circ}$ and the processing time are specified. For the above couples (L, s) the closedness of $\Sigma(L, s)$ [which was proved in Ovchinnikov (1992) for all $L \geq 2$ and $s \geq 3$] has been confirmed.

MINRE. Given a finite OMP E with a full set S of two-valued states, the MINRE finds all minimal full subsets of S (and thus all minimal representations for E with all point states contained in S). Also, additionally given a subgroup of the automorphism group of E, the MINRE can find the representations up to the subgroup. To do this, in the data file, ***.dmr**, the user should specify the card$S$, the number of atoms in E, and the atoms themselves. The user may (if he or she knows) specify a collection of automorphisms for the corresponding representation. After stopping to run the MINRE outputs the results file ***.mir.** To test the routine, some results of Sultanbekov (1993) have been confirmed. In particular, the number of all minimal representations, up to the group of all automorphisms, for E_6 has been confirmed to be equal to 10.

4. NEW RESULTS OBTAINED WITH GENER, BIPOLAR, AND MINRE

Let $\Lambda \subset \mathbf{R}^2$ be finite. Denote by $\Delta(\Lambda)$ the c.l. on Λ generated by all subsets of the following form: π_1^{-1} $(\{r\})$, π_2^{-1} $(\{r\})$ or $(\pi_1 + \pi_2)^{-1}$ $(\{r\})$, where $\pi_1(x, y) = x$, $\pi_2(x,y) = y$ $((x, y) \in \Lambda)$, and $r \in \mathbf{R}$.

We have made use of GENER together with BIPOLAR to establish the closedness for all $\Delta_n = \Delta(\Lambda)$, where $\Lambda = \{0,1, \ldots, n-1\}^2$ and $1 \leq n \leq 10$. For every such n, all atoms (in particular, their number, A_n) and all blocks (in particular, their number, B_n) have been found. Denote by M_n (resp., m_n) the maximal (resp., minimal) number of atoms in a block of Δ_n. Let C_n stand for the maximal number of elements of an atom in Δ_n. The numbers are given in Table I. A special program has been developed to compare the sets of atoms in $\mathscr{E}$ and $\mathscr{E}^{\infty}$.

The MINRE has enabled us to find all minimal representations for E_7. Their number (up to the group of all automorphisms) turns out to be equal to 546. For every $k \in \mathbf{N}$, Table II contains the number $N(k)$ of all minimal representations $(\Omega(k), \mathscr{E}(k))$ for E_7 with card$\Omega(k) = k$ up to the group of all automorphisms.

Table I

n	A_n	B_n	C_n	M_n	m_n	n	A_n	B_n	C_n	M_n	m_n
1	1	1	1	1	1	6	191	382	16	11	5
2	4	1	1	4	4	7	357	726	23	13	5
3	12	6	2	6	6	8	621	1307	32	15	5
4	39	60	5	7	5	9	1017	2154	44	17	5
5	91	165	10	9	5	10	1571	3356	52	19	5

Table II

k	$N(k)$	k	$N(k)$
10	1	15	130
11	2	16	89
12	45	17	7
13	84	Other	0
14	188		

For each of the 10 minimal representations for E_6 mentioned in Section 3, their bipolars have been found with BIPOLAR.

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